Modular Zilber-Pink with Derivatives

Vahagn Aslanyan

University of Manchester

2 July 2025

The j-function

- Let $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the complex upper half-plane.

The j-function

- Let $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the complex upper half-plane.
- $\mathrm{GL}_2^+(\mathbb{R})$ is the group of 2×2 matrices with real entries and positive determinant. It acts on $\mathbb{H}$ via linear fractional transformations. That is, for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$ we define

$$gz = \frac{az+b}{cz+d}.$$

- Let $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the complex upper half-plane.
- $\mathrm{GL}_2^+(\mathbb{R})$ is the group of 2×2 matrices with real entries and positive determinant. It acts on $\mathbb{H}$ via linear fractional transformations. That is, for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$ we define

$$gz = \frac{az+b}{cz+d}.$$

- The function $j : \mathbb{H} \to \mathbb{C}$ is a modular function of weight 0 for the modular group $\mathrm{SL}_2(\mathbb{Z})$ defined and analytic on $\mathbb{H}$.

The j-function

- Let $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the complex upper half-plane.
- $\mathrm{GL}_2^+(\mathbb{R})$ is the group of 2×2 matrices with real entries and positive determinant. It acts on $\mathbb{H}$ via linear fractional transformations. That is, for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$ we define

$$gz = \frac{az + b}{cz + d}.$$

- The function $j : \mathbb{H} \to \mathbb{C}$ is a modular function of weight 0 for the modular group $\mathrm{SL}_2(\mathbb{Z})$ defined and analytic on $\mathbb{H}$.
- $j(gz) = j(z)$ for all $g \in \mathrm{SL}_2(\mathbb{Z})$.

The j-function

- Let $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the complex upper half-plane.
- $\mathrm{GL}_2^+(\mathbb{R})$ is the group of 2×2 matrices with real entries and positive determinant. It acts on $\mathbb{H}$ via linear fractional transformations. That is, for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$ we define

$$gz = \frac{az + b}{cz + d}.$$

- The function $j : \mathbb{H} \to \mathbb{C}$ is a modular function of weight 0 for the modular group $\mathrm{SL}_2(\mathbb{Z})$ defined and analytic on $\mathbb{H}$.
- $j(gz) = j(z)$ for all $g \in \mathrm{SL}_2(\mathbb{Z})$.
- By means of j the quotient $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathbb{H}$ is identified with $\mathbb{C}$ (thus, j is a bijection from the fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$ to $\mathbb{C}$).

Modular polynomials

- There is a countable collection of irreducible polynomials $\Phi_N \in \mathbb{Z}[X,Y]$ ($N \geq 1$) such that for any $z_1, z_2 \in \mathbb{H}$

$$\Phi_N(j(z_1), j(z_2)) = 0 \text{ for some } N \text{ iff } z_2 = gz_1 \text{ for some } g \in \mathrm{GL}_2^+(\mathbb{Q}).$$

Modular polynomials

- There is a countable collection of irreducible polynomials $\Phi_N \in \mathbb{Z}[X, Y]$ $(N \geq 1)$ such that for any $z_1, z_2 \in \mathbb{H}$

$$\Phi_N(j(z_1), j(z_2)) = 0 \text{ for some } N \text{ iff } z_2 = gz_1 \text{ for some } g \in \mathrm{GL}_2^+(\mathbb{Q}).$$

- The polynomials Φ_N are called modular polynomials.

Modular polynomials

- There is a countable collection of irreducible polynomials $\Phi_N \in \mathbb{Z}[X, Y]$ ($N \geq 1$) such that for any $z_1, z_2 \in \mathbb{H}$

$$\Phi_N(j(z_1), j(z_2)) = 0 \text{ for some } N \text{ iff } z_2 = gz_1 \text{ for some } g \in \mathrm{GL}_2^+(\mathbb{Q}).$$

- The polynomials Φ_N are called modular polynomials.
- Two elements $w_1, w_2 \in \mathbb{C}$ (or from *any* field) are called modularly independent if they do not satisfy any modular relation $\Phi_N(w_1, w_2) = 0$.

j-special varieties

Definition

A j-special subvariety of $\mathbb{C}^n$ (coordinatised by $\bar{y}$) is an irreducible component of a variety defined by modular equations, i.e. equations of the form $\Phi_N(y_i, y_k) = 0$ for some $1 \leq i, k \leq n$ where $\Phi_N(X, Y)$ is a modular polynomial.

j-special varieties

Definition

A j-special subvariety of $\mathbb{C}^n$ (coordinatised by $\bar{y}$) is an irreducible component of a variety defined by modular equations, i.e. equations of the form $\Phi_N(y_i, y_k) = 0$ for some $1 \leq i, k \leq n$ where $\Phi_N(X, Y)$ is a modular polynomial.

Definition

A subvariety $U \subseteq \mathbb{H}^n$ (i.e. an intersection of $\mathbb{H}^n$ with some algebraic variety) is called $\mathbb{H}$-special if it is defined by some equations of the form $z_i = g_{i,k} z_k, \ i \neq k$, with $g_{i,k} \in \mathrm{GL}_2^+(\mathbb{Q})$, and some equations of the form $z_i = \tau_i$ where $\tau_i \in \mathbb{H}$ is a quadratic number.

j-special varieties

Definition

A j-special subvariety of $\mathbb{C}^n$ (coordinatised by $\bar{y}$) is an irreducible component of a variety defined by modular equations, i.e. equations of the form $\Phi_N(y_i, y_k) = 0$ for some $1 \leq i, k \leq n$ where $\Phi_N(X, Y)$ is a modular polynomial.

Definition

A subvariety $U \subseteq \mathbb{H}^n$ (i.e. an intersection of $\mathbb{H}^n$ with some algebraic variety) is called $\mathbb{H}$-special if it is defined by some equations of the form $z_i = g_{i,k} z_k, \ i \neq k$, with $g_{i,k} \in \mathrm{GL}_2^+(\mathbb{Q})$, and some equations of the form $z_i = \tau_i$ where $\tau_i \in \mathbb{H}$ is a quadratic number.
For such a U the image $j(U)$ is j-special (j is identified with its Cartesian powers).

Definition

For a variety $V \subseteq \mathbb{C}^n$ and a j-special variety $S \subseteq \mathbb{C}^n$, a component X of the intersection $V \cap S$ is a j-atypical subvariety of V if

$$\dim X > \dim V + \dim S - n.$$

Modular Zilber-Pink without Derivatives

Definition

For a variety $V \subseteq \mathbb{C}^n$ and a j-special variety $S \subseteq \mathbb{C}^n$, a component X of the intersection $V \cap S$ is a j-atypical subvariety of V if

$$\dim X > \dim V + \dim S - n.$$

Definition

Let $V \subseteq \mathbb{C}^n$ be an algebraic variety. We define the atypical set of V, denoted $\mathrm{Atyp}_j(V)$, as the union of all j-atypical subvarieties of V.

Modular Zilber-Pink without Derivatives

Definition

For a variety $V \subseteq \mathbb{C}^n$ and a j-special variety $S \subseteq \mathbb{C}^n$, a component X of the intersection $V \cap S$ is a j-atypical subvariety of V if

$$\dim X > \dim V + \dim S - n.$$

Definition

Let $V \subseteq \mathbb{C}^n$ be an algebraic variety. We define the atypical set of V, denoted $\mathrm{Atyp}_j(V)$, as the union of all j-atypical subvarieties of V.

Conjecture (Modular Zilber–Pink)

$\mathrm{Atyp}_j(V)$ *is contained in a finite union of proper j-special subvarieties of* $\mathbb{C}^n$*.*

Definition

A j-atypical subvariety X of $V \subseteq \mathbb{C}^n$ is strongly j-atypical if no coordinate is constant on X.

Definition

A j-atypical subvariety X of $V \subseteq \mathbb{C}^n$ is strongly j-atypical if no coordinate is constant on X.

Let $\mathrm{SAtyp}_j(V)$ denote the union of all strongly j-atypical subvarieties of V.

Weak Modular Zilber-Pink without Derivatives

Definition

A j-atypical subvariety X of $V \subseteq \mathbb{C}^n$ is strongly j-atypical if no coordinate is constant on X.
Let $\mathrm{SAtyp}_j(V)$ denote the union of all strongly j-atypical subvarieties of V.

Theorem (Pila-Tsimerman, 2015)

$\mathrm{SAtyp}_j(V)$ *is contained in a finite union of proper j-special subvarieties of $\mathbb{C}^n$.*

Weak Modular Zilber-Pink without Derivatives

Definition

A j-atypical subvariety X of $V \subseteq \mathbb{C}^n$ is strongly j-atypical if no coordinate is constant on X.
Let $\mathrm{SAtyp}_j(V)$ denote the union of all strongly j-atypical subvarieties of V.

Theorem (Pila-Tsimerman, 2015)

$\mathrm{SAtyp}_j(V)$ *is contained in a finite union of proper j-special subvarieties of $\mathbb{C}^n$.*

The proof is based on the Ax-Schanuel theorem for the j-function (due to Pila and Tsimerman).

J-special varieties

- Define a function $J : \mathbb{H} \to \mathbb{C}^3$ by
$$J : z \mapsto (j(z), j'(z), j''(z)).$$

J-special varieties

- Define a function $J : \mathbb{H} \to \mathbb{C}^3$ by
$$J : z \mapsto (j(z), j'(z), j''(z)).$$
- We extend J to $\mathbb{H}^n$ and define $J : \mathbb{H}^n \to \mathbb{C}^{3n}$ as follows:
$$J : \bar{z} \mapsto (j(\bar{z}), j'(\bar{z}), j''(\bar{z}))$$
where $\bar{z} := (z_1, \ldots, z_n)$ and $j^{(k)}(\bar{z}) = (j^{(k)}(z_1), \ldots, j^{(k)}(z_n))$ for $k = 0, 1, 2$.

J-special varieties

- Define a function $J : \mathbb{H} \to \mathbb{C}^3$ by
$$J : z \mapsto (j(z), j'(z), j''(z)).$$

- We extend J to $\mathbb{H}^n$ and define $J : \mathbb{H}^n \to \mathbb{C}^{3n}$ as follows:
$$J : \bar{z} \mapsto (j(\bar{z}), j'(\bar{z}), j''(\bar{z}))$$
where $\bar{z} := (z_1, \ldots, z_n)$ and $j^{(k)}(\bar{z}) = (j^{(k)}(z_1), \ldots, j^{(k)}(z_n))$ for $k = 0, 1, 2$.
- Note that $j'''(z)$ is algebraic over j, j', j'', so we do not need to consider third or higher derivatives.

J-special varieties

- Define a function $J : \mathbb{H} \to \mathbb{C}^3$ by

$$J : z \mapsto (j(z), j'(z), j''(z)).$$

- We extend J to $\mathbb{H}^n$ and define $J : \mathbb{H}^n \to \mathbb{C}^{3n}$ as follows:

$$J : \bar{z} \mapsto (j(\bar{z}), j'(\bar{z}), j''(\bar{z}))$$

where $\bar{z} := (z_1, \ldots, z_n)$ and $j^{(k)}(\bar{z}) = (j^{(k)}(z_1), \ldots, j^{(k)}(z_n))$ for $k = 0, 1, 2$.

- Note that $j'''(z)$ is algebraic over j, j', j'', so we do not need to consider third or higher derivatives.

Definition (Pila)

Let $U \subseteq \mathbb{H}^n$ be $\mathbb{H}$-special. We denote by $\langle\langle U \rangle\rangle \subseteq \mathbb{C}^{3n}$ the Zariski closure of $J(U)$ over $\mathbb{Q}^{\text{alg}}$. These are the J-special varieties in $\mathbb{C}^{3n}$.

J-special varieties

- Define a function $J : \mathbb{H} \to \mathbb{C}^3$ by
$$J : z \mapsto (j(z), j'(z), j''(z)).$$
- We extend J to $\mathbb{H}^n$ and define $J : \mathbb{H}^n \to \mathbb{C}^{3n}$ as follows:
$$J : \bar{z} \mapsto (j(\bar{z}), j'(\bar{z}), j''(\bar{z}))$$
where $\bar{z} := (z_1, \ldots, z_n)$ and $j^{(k)}(\bar{z}) = (j^{(k)}(z_1), \ldots, j^{(k)}(z_n))$ for $k = 0, 1, 2$.
- Note that $j'''(z)$ is algebraic over j, j', j'', so we do not need to consider third or higher derivatives.

Definition (Pila)

Let $U \subseteq \mathbb{H}^n$ be $\mathbb{H}$-special. We denote by $\langle\langle U \rangle\rangle \subseteq \mathbb{C}^{3n}$ the Zariski closure of $J(U)$ over $\mathbb{Q}^{\mathrm{alg}}$. These are the J-special varieties in $\mathbb{C}^{3n}$.

Remark

J-special varieties are irreducible, and can be defined purely algebraically.

Definition

For a variety $V \subseteq \mathbb{C}^{3n}$ we let the *J-atypical set* of V, denoted $\mathrm{Atyp}_J(V)$, be the union of all atypical components of intersections $V \cap T$ in $\mathbb{C}^{3n}$ where $T \subseteq \mathbb{C}^{3n}$ is a J-special variety.

Definition

For a variety $V \subseteq \mathbb{C}^{3n}$ we let the J-atypical set of V, denoted $\mathrm{Atyp}_J(V)$, be the union of all atypical components of intersections $V \cap T$ in $\mathbb{C}^{3n}$ where $T \subseteq \mathbb{C}^{3n}$ is a J-special variety.

Conjecture (Pila, "MZPD")

For every algebraic variety $V \subseteq \mathbb{C}^{3n}$ there is a finite collection Σ of proper $\mathbb{H}$-special subvarieties of $\mathbb{H}^n$ such that

$$\mathrm{Atyp}_J(V) \cap J(\mathbb{H}^n) \subseteq \bigcup_{\substack{U \in \Sigma \\ \bar{\gamma} \in \mathrm{SL}_2(\mathbb{Z})^n}} \langle\langle \bar{\gamma} U \rangle\rangle.$$

Modular Zilber-Pink with Derivatives

Definition

For a variety $V \subseteq \mathbb{C}^{3n}$ we let the J-atypical set of V, denoted $\mathrm{Atyp}_J(V)$, be the union of all atypical components of intersections $V \cap T$ in $\mathbb{C}^{3n}$ where $T \subseteq \mathbb{C}^{3n}$ is a J-special variety.

Conjecture (Pila, "MZPD")

For every algebraic variety $V \subseteq \mathbb{C}^{3n}$ there is a finite collection Σ of proper $\mathbb{H}$-special subvarieties of $\mathbb{H}^n$ such that

$$\mathrm{Atyp}_J(V) \cap J(\mathbb{H}^n) \subseteq \bigcup_{\substack{U \in \Sigma \\ \bar{\gamma} \in \mathrm{SL}_2(\mathbb{Z})^n}} \langle\langle \bar{\gamma} U \rangle\rangle.$$

Remark

Here we may need infinitely many J-special subvarieties to cover the atypical set of V, but the collection is conjectured to be "finitely generated".

Weak Modular Zilber-Pink with Derivatives

Definition

For a J-special variety $T \subseteq \mathbb{C}^{3n}$ and an algebraic variety $V \subseteq \mathbb{C}^{3n}$ an atypical component X of an intersection $V \cap T$ in $\mathbb{C}^{3n}$ is a strongly J-atypical subvariety of V if for every irreducible analytic component Y of $X \cap J(\mathbb{H}^n)$, no coordinate is constant on Y.

Definition

For a J-special variety $T \subseteq \mathbb{C}^{3n}$ and an algebraic variety $V \subseteq \mathbb{C}^{3n}$ an atypical component X of an intersection $V \cap T$ in $\mathbb{C}^{3n}$ is a strongly J-atypical subvariety of V if for every irreducible analytic component Y of $X \cap J(\mathbb{H}^n)$, no coordinate is constant on Y.

The strongly J-atypical set of V, denoted $\mathrm{SAtyp}_J(V)$, is the union of all strongly J-atypical subvarieties of V.

Weak Modular Zilber-Pink with Derivatives

Definition

For a J-special variety $T \subseteq \mathbb{C}^{3n}$ and an algebraic variety $V \subseteq \mathbb{C}^{3n}$ an atypical component X of an intersection $V \cap T$ in $\mathbb{C}^{3n}$ is a strongly J-atypical subvariety of V if for every irreducible analytic component Y of $X \cap J(\mathbb{H}^n)$, no coordinate is constant on Y.
The strongly J-atypical set of V, denoted $\mathrm{SAtyp}_J(V)$, is the union of all strongly J-atypical subvarieties of V.

Theorem (A., 2021)

For every algebraic variety $V \subseteq \mathbb{C}^{3n}$ there is a finite collection Σ of proper $\mathbb{H}$-special subvarieties of $\mathbb{H}^n$ such that

$$\mathrm{SAtyp}_J(V) \cap J(\mathbb{H}^n) \subseteq \bigcup_{\substack{U \in \Sigma \\ \bar{\gamma} \in \mathrm{SL}_2(\mathbb{Z})^n}} \langle\langle \bar{\gamma} U \rangle\rangle.$$

Sketch of Proof - Complex Ax-Schanuel

- Let $\Gamma \subseteq \mathbb{H}^n \times \mathbb{C}^{3n}$ be the graph of $J : \mathbb{H}^n \to \mathbb{C}^{3n}$.

Sketch of Proof - Complex Ax-Schanuel

- Let $\Gamma \subseteq \mathbb{H}^n \times \mathbb{C}^{3n}$ be the graph of $J : \mathbb{H}^n \to \mathbb{C}^{3n}$.
- Let $\mathrm{pr}_j : \mathbb{C}^{4n} \to \mathbb{C}^n$ be the projection to the j-coordinates, i.e. the second n coordinates.

Sketch of Proof - Complex Ax-Schanuel

- Let $\Gamma \subseteq \mathbb{H}^n \times \mathbb{C}^{3n}$ be the graph of $J : \mathbb{H}^n \to \mathbb{C}^{3n}$.
- Let $\mathrm{pr}_j : \mathbb{C}^{4n} \to \mathbb{C}^n$ be the projection to the j-coordinates, i.e. the second n coordinates.

Theorem (Complex Ax-Schanuel for j, Pila-Tsimerman 2015)

Let $V \subseteq \mathbb{C}^{4n}$ be an algebraic variety and let A be an analytic component of the intersection $V \cap \Gamma$. If $\dim A > \dim V - 3n$ and no coordinate is constant on $\mathrm{pr}_j A$ then it is contained in a proper j-special subvariety of $\mathbb{C}^n$.

Sketch of Proof - Complex Ax-Schanuel

- Let $\Gamma \subseteq \mathbb{H}^n \times \mathbb{C}^{3n}$ be the graph of $J : \mathbb{H}^n \to \mathbb{C}^{3n}$.
- Let $\mathrm{pr}_j : \mathbb{C}^{4n} \to \mathbb{C}^n$ be the projection to the j-coordinates, i.e. the second n coordinates.

Theorem (Complex Ax-Schanuel for j, Pila-Tsimerman 2015)

Let $V \subseteq \mathbb{C}^{4n}$ be an algebraic variety and let A be an analytic component of the intersection $V \cap \Gamma$. If $\dim A > \dim V - 3n$ and no coordinate is constant on $\mathrm{pr}_j A$ then it is contained in a proper j-special subvariety of $\mathbb{C}^n$.

Theorem (Uniform Ax-Schanuel)

Let $V_{\bar{c}} \subseteq \mathbb{C}^{4n}$ be a parametric family of algebraic varieties. Then there is a finite collection Σ of proper j-special subvarieties of $\mathbb{C}^n$ such that for every $\bar{c} \subseteq \mathbb{C}$, if $A_{\bar{c}}$ is an analytic component of the intersection $V_{\bar{c}} \cap \Gamma$ with $\dim A_{\bar{c}} > \dim V_{\bar{c}} - 3n$, and no coordinate is constant on $\mathrm{pr}_j A_{\bar{c}}$, then $\mathrm{pr}_j A_{\bar{c}}$ is contained in some $T' \in \Sigma$.

Sketch of Proof - Complex Ax-Schanuel

- Let $\Gamma \subseteq \mathbb{H}^n \times \mathbb{C}^{3n}$ be the graph of $J : \mathbb{H}^n \to \mathbb{C}^{3n}$.
- Let $\mathrm{pr}_j : \mathbb{C}^{4n} \to \mathbb{C}^n$ be the projection to the j-coordinates, i.e. the second n coordinates.

Theorem (Complex Ax-Schanuel for j, Pila-Tsimerman 2015)

Let $V \subseteq \mathbb{C}^{4n}$ be an algebraic variety and let A be an analytic component of the intersection $V \cap \Gamma$. If $\dim A > \dim V - 3n$ and no coordinate is constant on $\mathrm{pr}_j A$ then it is contained in a proper j-special subvariety of $\mathbb{C}^n$.

Theorem (Uniform Ax-Schanuel)

Let $V_{\bar{c}} \subseteq \mathbb{C}^{4n}$ be a parametric family of algebraic varieties. Then there is a finite collection Σ of proper j-special subvarieties of $\mathbb{C}^n$ such that for every $\bar{c} \subseteq \mathbb{C}$, if $A_{\bar{c}}$ is an analytic component of the intersection $V_{\bar{c}} \cap \Gamma$ with $\dim A_{\bar{c}} > \dim V_{\bar{c}} - 3n$, and no coordinate is constant on $\mathrm{pr}_j A_{\bar{c}}$, then $\mathrm{pr}_j A_{\bar{c}}$ is contained in some $T' \in \Sigma$.

The proof is based on the compactness theorem of first-order logic.

Sketch of proof of Weak Modular ZP with Derivatives

- Let $T = \langle\langle U \rangle\rangle \subseteq \mathbb{C}^{3n}$ be a J-special variety and $X \subseteq V \cap T$ be a strongly atypical component: $\dim X > \dim V + \dim T - 3n$.

Sketch of proof of Weak Modular ZP with Derivatives

- Let $T = \langle\langle U \rangle\rangle \subseteq \mathbb{C}^{3n}$ be a J-special variety and $X \subseteq V \cap T$ be a strongly atypical component: $\dim X > \dim V + \dim T - 3n$.
- Assume $A \subseteq X \cap J(\mathbb{H}^n)$ is an analytic component such that no coordinate is constant on A. Then $A \subseteq J(U) \subseteq T$, and A is an analytic component of $X \cap J(U)$.

Sketch of proof of Weak Modular ZP with Derivatives

- Let $T = \langle\langle U \rangle\rangle \subseteq \mathbb{C}^{3n}$ be a J-special variety and $X \subseteq V \cap T$ be a strongly atypical component: $\dim X > \dim V + \dim T - 3n$.
- Assume $A \subseteq X \cap J(\mathbb{H}^n)$ is an analytic component such that no coordinate is constant on A. Then $A \subseteq J(U) \subseteq T$, and A is an analytic component of $X \cap J(U)$.
- Then

$$\dim A \geq \dim X + \dim J(U) - \dim T >$$
$$\dim V + \dim T - 3n + \dim J(U) - \dim T = \dim V + \dim U - 3n.$$

Sketch of proof of Weak Modular ZP with Derivatives

- Let $T = \langle\langle U \rangle\rangle \subseteq \mathbb{C}^{3n}$ be a J-special variety and $X \subseteq V \cap T$ be a strongly atypical component: $\dim X > \dim V + \dim T - 3n$.
- Assume $A \subseteq X \cap J(\mathbb{H}^n)$ is an analytic component such that no coordinate is constant on A. Then $A \subseteq J(U) \subseteq T$, and A is an analytic component of $X \cap J(U)$.
- Then

$$\dim A \geq \dim X + \dim J(U) - \dim T >$$
$$\dim V + \dim T - 3n + \dim J(U) - \dim T = \dim V + \dim U - 3n.$$

- This implies

$$\dim((U \times A) \cap \Gamma) = \dim A > \dim(U \times V) - 3n.$$

Sketch of proof of Weak Modular ZP with Derivatives

- Let $T = \langle\langle U \rangle\rangle \subseteq \mathbb{C}^{3n}$ be a J-special variety and $X \subseteq V \cap T$ be a strongly atypical component: $\dim X > \dim V + \dim T - 3n$.
- Assume $A \subseteq X \cap J(\mathbb{H}^n)$ is an analytic component such that no coordinate is constant on A. Then $A \subseteq J(U) \subseteq T$, and A is an analytic component of $X \cap J(U)$.
- Then

$$\dim A \geq \dim X + \dim J(U) - \dim T >$$
$$\dim V + \dim T - 3n + \dim J(U) - \dim T = \dim V + \dim U - 3n.$$

- This implies

$$\dim((U \times A) \cap \Gamma) = \dim A > \dim(U \times V) - 3n.$$

- Now the desired result follows from Uniform Ax-Schanuel applied to the parametric family of algebraic varieties $W_{\bar{c}} \times V$ where $W_{\bar{c}}$ varies over the parametric family of all $\mathbb{C}$-geodesic varieties defined by $\mathrm{GL}_2(\mathbb{C})$-equations.

Connection to Existential Closedness

- In fact, the above argument proves a slightly stronger (but "more analytic") version of the theorem.

Connection to Existential Closedness

- In fact, the above argument proves a slightly stronger (but "more analytic") version of the theorem.
- By applying the argument to the class of functions $J(gz)$ where $g \in \mathrm{GL}_2(\mathbb{C})$ we can establish a much stronger result, which then we can translate into a differential algebraic language (using Seidenberg's embedding theorem).

Connection to Existential Closedness

- In fact, the above argument proves a slightly stronger (but "more analytic") version of the theorem.
- By applying the argument to the class of functions $J(gz)$ where $g \in \mathrm{GL}_2(\mathbb{C})$ we can establish a much stronger result, which then we can translate into a differential algebraic language (using Seidenberg's embedding theorem).
- The above-mentioned differential algebraic result also has a differential algebraic proof which uses Differential Existential Closedness for J.

Connection to Existential Closedness

- In fact, the above argument proves a slightly stronger (but "more analytic") version of the theorem.
- By applying the argument to the class of functions $J(gz)$ where $g \in \mathrm{GL}_2(\mathbb{C})$ we can establish a much stronger result, which then we can translate into a differential algebraic language (using Seidenberg's embedding theorem).
- The above-mentioned differential algebraic result also has a differential algebraic proof which uses Differential Existential Closedness for J.
- If we could state a fully algebraic version of Modular ZP with Derivatives, it would then be more amenable to algebraic geometric and differential algebraic methods.

Connection to Existential Closedness

- In fact, the above argument proves a slightly stronger (but "more analytic") version of the theorem.
- By applying the argument to the class of functions $J(gz)$ where $g \in \mathrm{GL}_2(\mathbb{C})$ we can establish a much stronger result, which then we can translate into a differential algebraic language (using Seidenberg's embedding theorem).
- The above-mentioned differential algebraic result also has a differential algebraic proof which uses Differential Existential Closedness for J.
- If we could state a fully algebraic version of Modular ZP with Derivatives, it would then be more amenable to algebraic geometric and differential algebraic methods.
- To that end we need to understand which algebraic varieties $V \subseteq \mathbb{C}^{3n}$ intersect the image of J, i.e. $J(\mathbb{H}^n)$.

Connection to Existential Closedness

- In fact, the above argument proves a slightly stronger (but "more analytic") version of the theorem.
- By applying the argument to the class of functions $J(gz)$ where $g \in \mathrm{GL}_2(\mathbb{C})$ we can establish a much stronger result, which then we can translate into a differential algebraic language (using Seidenberg's embedding theorem).
- The above-mentioned differential algebraic result also has a differential algebraic proof which uses Differential Existential Closedness for J.
- If we could state a fully algebraic version of Modular ZP with Derivatives, it would then be more amenable to algebraic geometric and differential algebraic methods.
- To that end we need to understand which algebraic varieties $V \subseteq \mathbb{C}^{3n}$ intersect the image of J, i.e. $J(\mathbb{H}^n)$.
- This is a special case of the Existential Closedness Conjecture stating that *broad* and *free* varieties intersect the graph/image of J.

Connection to Existential Closedness

- In fact, the above argument proves a slightly stronger (but "more analytic") version of the theorem.
- By applying the argument to the class of functions $J(gz)$ where $g \in \mathrm{GL}_2(\mathbb{C})$ we can establish a much stronger result, which then we can translate into a differential algebraic language (using Seidenberg's embedding theorem).
- The above-mentioned differential algebraic result also has a differential algebraic proof which uses Differential Existential Closedness for J.
- If we could state a fully algebraic version of Modular ZP with Derivatives, it would then be more amenable to algebraic geometric and differential algebraic methods.
- To that end we need to understand which algebraic varieties $V \subseteq \mathbb{C}^{3n}$ intersect the image of J, i.e. $J(\mathbb{H}^n)$.
- This is a special case of the Existential Closedness Conjecture stating that *broad* and *free* varieties intersect the graph/image of J.
- If Existential Closedness holds then Modular ZP with Derivatives can be rephrased in terms of those J-atypical subvarieties which are broad and free (these are algebraic conditions).